SCHWARZIAN DERIVATIVE CRITERIA FOR VALENCE OF ANALYTIC AND HARMONIC MAPPINGS

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Abstract. For analytic functions in the unit disk, general bounds on the Schwarzian derivative in terms of Nehari functions are shown to imply uniform local univalence and in some cases finite and bounded valence. Similar results are obtained for the Weierstrass–Enneper lifts of planar harmonic mappings to their associated minimal surfaces. Finally, certain classes of harmonic mappings are shown to have finite Schwarzian norm.

§1. Introduction.

The *Schwarzian derivative* of an analytic locally univalent function f is defined by

$$\mathcal{S}f = (f''/f')' - \tfrac{1}{2}(f''/f')^2 .$$

Its most important property is Möbius invariance: $\mathcal{S}(T \circ f) = \mathcal{S}f$ for every Möbius transformation

$$T(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 .$$

More generally, if g is any function analytic and locally univalent on the range of f, then

$$\mathcal{S}(g \circ f) = ((\mathcal{S}g) \circ f){f'}^2 + \mathcal{S}f .$$

As a special case, $\mathcal{S}(g \circ T) = ((\mathcal{S}g) \circ T){T'}^2$, since $\mathcal{S}T = 0$ for every Möbius transformation T. For an arbitrary analytic function ψ, the functions f with Schwarzian $\mathcal{S}f = 2\psi$ are those of the form $f = w_1/w_2$, where w_1 and w_2 are linearly independent solutions of the linear differential equation $w'' + \psi w = 0$. It follows that $\mathcal{S}f = \mathcal{S}g$ implies $f = T \circ g$ for some Möbius transformation T. In particular, Möbius transformations are the only functions with $\mathcal{S}f = 0$.

Nehari [15] found that certain estimates on the Schwarzian imply global univalence. Specifically, he showed that if f is analytic and locally univalent in the unit disk $\mathbb{D}$ and its Schwarzian satisfies either $|\mathcal{S}f(z)| \le 2(1-|z|^2)^{-2}$ or $|\mathcal{S}f(z)| \le \pi^2/2$ for all $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$. Pokornyi [19] then stated, and Nehari [16] proved, that the condition $|\mathcal{S}f(z)| \le 4(1-|z|^2)^{-1}$ also implies univalence. In fact, Nehari [16] unified all three criteria by proving that f is univalent under the

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general hypothesis $|\mathcal{S}f(z)| \le 2p(|z|)$, where $p(x)$ is a positive continuous even function with the properties that $(1-x^2)^2 p(x)$ is nonincreasing on the interval $[0,1)$ and no nontrivial solution u of the differential equation $u'' + pu = 0$ has more than one zero in $(-1,1)$. Equivalently, the last condition can be replaced by the requirement that some solution of the differential equation has no zeros in $(-1,1)$. We will refer to such functions $p(x)$ as *Nehari functions*. It can be seen that the functions $p(x) = (1-x^2)^{-2}$, $p(x) = \pi^2/4$, and $p(x) = 2(1-x^2)^{-1}$ are all Nehari functions; the nonvanishing solutions are respectively $u = \sqrt{1-x^2}$, $u = \cos(\pi x/2)$, and $u = 1-x^2$.

The constant 2 in Nehari's general criterion is sharp. In many important cases, when $C > 2$ the inequality $|\mathcal{S}f(z)| \le C\,p(|z|)$ admits nonunivalent functions f. For $C > 2$ and $p(x) = (1-x^2)^{-2}$, Hille [11] gave an explicit example of an analytic function f satisfying $|\mathcal{S}f(z)| \le C(1-|z|^2)^{-2}$, yet having infinite valence in the unit disk. However, this condition does imply *uniform* local univalence in the sense that any two points z_1 and z_2 where $f(z_1) = f(z_2)$ are separated by a certain minimum distance in the hyperbolic metric. Further details and generalizations will appear in the next section.

In previous work [3] we introduced a notion of Schwarzian derivative for harmonic mappings, or complex-valued harmonic functions. In [4] we generalized Nehari's univalence criterion to the (conformal) lift $\widetilde{f}$ of a harmonic mapping f to its associated minimal surface Σ. Under an inequality of the form

$$|\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \le 2p(|z|)\,, \qquad z \in \mathbb{D}\,, \tag{1}$$

for some Nehari function p, we showed that $\widetilde{f}$ is univalent in $\mathbb{D}$. Here K denotes the Gauss curvature of Σ and e^{σ} is the relevant conformal factor. When f is analytic, the surface Σ is the complex plane and $\widetilde{f} = f$, so that $K = 0$ and the condition (1) reduces to Nehari's criterion.

In the present paper we examine the consequences of a weaker inequality in which the right-hand side of (1) is replaced by $C\,p(|z|)$ for some constant $C > 2$. We begin by discussing the implications for analytic functions, where the valence of f depends crucially on the growth of $p(x)$ as $x \to 1$. This sets the stage for a more general study of the valence of harmonic lifts. The paper concludes with some remarks about Schwarzian norms of harmonic mappings.

§2. Valence of analytic functions.

The *valence* of a function f analytic in $\mathbb{D}$ is defined by $\sup_{w\in\mathbb{C}} n(f,w)$, where $n(f,w) \le \infty$ is the number of points $z \in \mathbb{D}$ for which $f(z) = w$.

For points $\alpha, \beta \in \mathbb{D}$, the hyperbolic metric is

$$d(\alpha,\beta) = \frac{1}{2}\,\log\frac{1+\rho(\alpha,\beta)}{1-\rho(\alpha,\beta)}\,, \qquad \text{where} \quad \rho(\alpha,\beta) = \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|\,.$$

Our point of departure is the following known theorem.

Theorem A. *Let f be analytic and locally univalent, and suppose that*

$$|\mathcal{S}f(z)| \leq \frac{2(1+\delta^2)}{(1-|z|^2)^2}\,, \qquad z \in \mathbb{D}\,,$$

for some constant $\delta > 0$. Then $d(\alpha,\beta) \geq \pi/\delta$ for any pair of points $\alpha, \beta \in \mathbb{D}$ where $f(\alpha) = f(\beta)$. The lower bound is best possible.

This result is essentialy due to B. Schwarz [21] and was further developed by Beesack and Schwarz [2]. It was rediscovered independently by Minda [14] and Overholt [18]. Here is a relatively simple proof, which will serve as a model for arguments given later in this paper.

Proof. Suppose $f(\alpha) = f(\beta)$ for $\alpha \neq \beta$ and consider the function $g = f \circ T$, where T is a Möbius self-mapping of the disk chosen so that $T(0) = \alpha$ and $T(b) = \beta$ for some $b > 0$. Then $g(0) = g(b)$ and $d(\alpha,\beta) = d(0,b)$ by the Möbius invariance of the hyperbolic metric. Combining the hypothesis on $\mathcal{S}f$ with the relation $\mathcal{S}g = ((\mathcal{S}f)\circ T){T'}^2$, we find after a calculation that

$$|\mathcal{S}g(\zeta)| \leq \frac{2(1+\delta^2)}{(1-|\zeta|^2)^2}\,, \qquad \zeta \in \mathbb{D}\,.$$

In other words, the bound on the Schwarzian is Möbius invariant.

Now let $q = \frac{1}{2}\mathcal{S}g$ and observe that the condition $g(0) = g(b)$ implies that some solution of the differential equation $u'' + qu = 0$ has the property $u(0) = u(b) = 0$. Without loss of generality, we may assume that $u(x) \neq 0$ for $0 < x < b$. Let $v = |u|$, or $v^2 = u\overline{u}$, so that

$$vv' = \mathrm{Re}\{\overline{u}u'\} \qquad \text{and} \qquad |v'(x)| \leq |u'(x)|\,.$$

Differentiating the first of these relations and applying the second, we find

$$v(x)v''(x) \geq \mathrm{Re}\{\overline{u(x)}u''(x)\} = -|u(x)|^2\mathrm{Re}\{q(x)\}\,,$$

which implies that $v''(x) + |q(x)|v(x) \geq 0$. Since $v(x) > 0$ in the interval $(0,b)$, it follows that v satisfies the differential equation $v'' + hv = 0$, where

$$h(x) = -\frac{v''(x)}{v(x)} \leq |q(x)| = \tfrac{1}{2}|\mathcal{S}g(x)| \leq \frac{1+\delta^2}{(1-x^2)^2}$$

for $0 < x < b$. By the Sturm comparison theorem, if $w(x)$ is a solution of the differential equation

$$w''(x) + \frac{1+\delta^2}{(1-x^2)^2}\,w(x) = 0\,, \tag{2}$$

with $w(0) = 0$, then $w(x)$ also vanishes at a point $c \in (0, b]$. But the solutions of (2) with $w(0) = 0$ are constant multiples of

$$w(x) = \sqrt{1 - x^2}\, \sin\left(\frac{\delta}{2} \log \frac{1+x}{1-x}\right) . \qquad (3)$$

(See Kamke [12], p. 492, eq. 2.369.) Thus $w(c) = 0$ for

$$\frac{\delta}{2} \log \frac{1+c}{1-c} = \pi\,, \qquad \text{or} \;\; d(0, c) = \frac{\pi}{\delta} .$$

Consequently,

$$d(\alpha, \beta) = d(0, b) \geq d(0, c) = \frac{\pi}{\delta} .$$

As $\delta \to 0$ this yields Nehari's univalence criterion with $p(x) = (1 - x^2)^{-2}$.

To see that the bound is sharp, consider the example

$$f(z) = \left(\frac{1+z}{1-z}\right)^{i\delta}, \qquad f(0) = 1\,,$$

given by Hille [10]. A calculation shows that

$$\mathcal{S}f(z) = \frac{2(1+\delta^2)}{(1-z^2)^2}\,, \quad \text{so that} \quad |\mathcal{S}f(z)| \leq \frac{2(1+\delta^2)}{(1-|z|^2)^2} .$$

The function f has infinite valence in the disk. For instance, $f(z) = 1$ whenever

$$\delta \log \frac{1+z}{1-z} = 2n\pi\,, \qquad n = 0, 1, 2, \ldots .$$

Defining x_n by the equation

$$\delta \log \frac{1+x_n}{1-x_n} = 2n\pi\,,$$

we see that $0 = x_0 < x_1 < x_2 < \cdots < 1$ and $d(x_n, x_{n+1}) = \pi/\delta$. Thus the lower bound π/δ is best possible. This concludes the proof.

We now turn to the more general case where

$$|\mathcal{S}f(z)| \leq C\, p(|z|)$$

for some Nehari function p and some constant $C > 2$. The valence of f then depends on the size of C and the growth of $p(x)$ as $x \to 1$. We begin with a general discussion of Nehari functions.

Since $(1-x^2)^2 p(x)$ is positive and nonincreasing on the interval $(0,1)$, the limit

$$\mu = \lim_{x\to 1-} (1-x^2)^2 p(x) \tag{4}$$

exists and $\mu \geq 0$. Observe first that $\mu \leq 1$. Indeed, if $\mu > 1$ then

$$p(x) > \frac{\frac{1}{2}(1+\mu)}{(1-x^2)^2}$$

in some interval $x_0 < x < 1$, and the Sturm comparison theorem shows that the solutions of the differential equation $u'' + pu = 0$ have infinitely many zeros in $(-1,1)$, which is not possible because p is a Nehari function.

If $\mu = 1$, we claim that $p(x)$ must be the function $(1-x^2)^{-2}$. The preceding argument does not quite capture this result. The proof will appeal instead to the following general extension of the Sturm theory, implicit in previous papers by Chuaqui and Osgood [7], Chuaqui and Gevirtz [6] and Chuaqui, Duren, and Osgood [4].

Relative Convexity Lemma. *Let P and Q be continuous functions on a finite interval $[a,b]$, with $Q(x) \leq P(x)$. Let u and v be solutions to the respective differential equations $u'' + Pu = 0$ and $v'' + Qv = 0$, with $u(a) = v(a) = 1$ and $u'(a) = v'(a) = 0$. Suppose that $u(x) > 0$ and $v(x) > 0$ in $[a,b)$ and define the function*

$$F(x) = \int_a^x v(t)^{-2}\, dt\,, \qquad a \leq x < b\,.$$

Then F is continuous and increasing on $[a,b)$, and it maps this interval onto an interval $[0,L)$, where $0 < L \leq \infty$. Let G denote the inverse of F. Then the function

$$w(y) = \frac{u(G(y))}{v(G(y))}$$

is concave on the interval $[0,L)$.

Proof. We are to show that $w''(y) \leq 0$. Differentiation gives

$$w' = \frac{vu' - uv'}{v^2}\, G' = vu' - uv'\,,$$

since $G'(y) = 1/F'(G(y)) = v(G(y))^2$. Another differentiation gives

$$w'' = (vu'' - uv'')G' = uv(Q-P)v^2 = (Q-P)v^4 w\,.$$

Since $Q(x) \leq P(x)$ and $w(y) > 0$ by hypothesis, this shows that $w''(y) \leq 0$ for $0 \leq y < L$.

With the lemma in hand, we now show that $p(x) = (1-x^2)^{-2}$ if $\mu = 1$. Because $(1-x^2)^2 p(x)$ decreases to 1, it is clear that $p(x) \ge (1-x^2)^{-2}$. Let $q(x) = (1-x^2)^{-2}$ and note that $v(x) = \sqrt{1-x^2}$ is the solution of $v'' + qv = 0$ on $[0,1)$ with $v(0) = 1$ and $v'(0) = 0$. Thus

$$F(x) = \int_0^x v(t)^{-2}\,dt = \frac{1}{2}\log\frac{1+x}{1-x} \to \infty \qquad \text{as } x \to 1\,,$$

and the inverse function $G(y)$ is defined on the interval $0 \le y < \infty$. If u is the solution of $u'' + pu = 0$ with $u(0) = 1$ and $u'(0) = 0$, the relative convexity lemma says that $w(y) = u(G(y))/v(G(y))$ is concave on $[0,\infty)$, with $w(0) = 1$ and $w'(0) = 0$. If $p(x) \not\equiv q(x)$, then $w(y) \not\equiv 1$ and so $0 < w(y_0) < 1$ and $w'(y_0) < 0$ for some point $y_0 \in (0,\infty)$. Since the curve $w(y)$ is concave, it lies below its tangent line constructed at y_0, and this line must cross the y-axis. Therefore, $w(y_1) = 0$ for some point $y_1 \in (y_0,\infty)$, which implies that $u(x_1) = 0$ for some point $x_1 \in (0,1)$. But this is not possible because by definition of a Nehari function the solution u cannot vanish more than once in $(-1,1)$. However, u is an even function since p is even, so $u(x_1) = 0$ would imply $u(-x_1) = 0$. This contradiction shows that $p(x) \equiv q(x) = (1-x^2)^{-2}$ in $[0,1)$, hence in $(-1,1)$.

Consequently, we may suppose that $0 \le \mu < 1$. It is relevant to note that each such value of μ actually arises from some Nehari function. For a parameter t in the interval $1 < t < 2$,

$$p(x) = \frac{t(1-(t-1)x^2)}{(1-x^2)^2}\,, \tag{5}$$

is a Nehari function with $\mu = t(2-t)$. Indeed, it can be checked that the function $u(x) = (1-x^2)^{t/2}$ solves the differential equation $u'' + pu = 0$ and has no zeros on $(-1,1)$. The other required properties of a Nehari function are easily verified, so this shows that each $\mu \in (0,1)$ arises from a Nehari function. The Nehari functions $p(x) = \pi^2/4$ and $p(x) = 2(1-x^2)^{-1}$, mentioned in the introduction, have $\mu = 0$.

Incidentally, it can be shown that for $1 < t < 2$ the function

$$f(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^t}$$

has Schwarzian derivative satisfying $|\mathcal{S}f(z)| \le 2p(|z|)$ for all $z \in \mathbb{D}$, where p is the Nehari function defined by (5). Thus by Nehari's theorem, f is univalent in $\mathbb{D}$, a result that Nehari [17] obtained by a different method.

The following theorem is an analogue of Theorem A for other Nehari functions.

Theorem 1. *Let f be analytic and locally univalent in $\mathbb{D}$, and suppose that*

$$|\mathcal{S}f(z)| \le C\,p(|z|)\,, \qquad z \in \mathbb{D}\,,$$

for some constant $C > 2$ and some Nehari function p with $0 \le \mu < 1$, where μ is defined by (4). Then f is uniformly locally univalent with respect to the hyperbolic metric. If $C\mu < 2$, the valence of f is finite and has a bound independent of f.

Corollary. *Let p be a Nehari function with $\mu = 0$, and suppose that*

$$|\mathcal{S}f(z)| \le C\,p(|z|)\,, \qquad z \in \mathbb{D}\,,$$

for some constant C. Then f has finite valence in $\mathbb{D}$.

In the statement of the theorem the case $\mu = 1$ is excluded because the uniform local univalence is then guaranteed by Theorem A. The case $C \le 2$ is excluded because the condition $|\mathcal{S}f(z)| \le 2\,p(|z|)$ is known to imply that f is univalent (Nehari's theorem).

Proof of Theorem 1. Because $\mu < 1$ it is clear that $p(x) \le (1-x^2)^{-2}$ for all x in some interval $r < x < 1$, and so

$$|\mathcal{S}f(z)| \le C\,p(|z|) \le \frac{C}{(1-|z|^2)^2}\,, \qquad r < |z| < 1\,.$$

Consequently, $|\mathcal{S}f(z)| \le C_1(1-|z|^2)^{-2}$ for some constant $C_1 \ge C$ and all $z \in \mathbb{D}$. Setting $C_1 = 2(1+\delta^2)$ with $\delta > 0$, one can infer from Theorem A that $d(\alpha,\beta) \ge \pi/\delta$ for any pair of points where $f(\alpha) = f(\beta)$. Thus f is uniformly locally univalent with respect to the hyperbolic metric.

If $C\mu < 2$, we can say more. Then in some annulus $r < |z| < 1$ we have

$$|\mathcal{S}f(z)| \le C\,p(|z|) \le \frac{C_1}{(1-|z|^2)^2}$$

for some constant $C_1 < 2$. By a theorem of Gehring and Pommerenke ([10], Theorem 4), this implies that f has finite valence in the annulus $r < |z| < 1$, and there is a bound on its valence that depends only on C and p, not on f. In the disk $|z| \le r$ the uniform local univalence again shows that f has finite valence with a bound independent of f. Thus the same is true in $\mathbb{D}$, as the theorem asserts. It should be remarked that a result slightly weaker than that of Gehring and Pommerenke is implicit in earlier work of B. Schwarz ([21], Theorem 1). The result of Schwarz shows that the condition $|\mathcal{S}f(z)| \le C(1-|z|^2)^{-2}$ implies $n(f,w) < \infty$ for each $w \in \mathbb{C}$ but does not provide a bound that is uniform in w.

As applications of Theorem 1, if $|\mathcal{S}f(z)| \le C$ or if $|\mathcal{S}f(z)| \le C(1-|z|^2)^{-1}$ in $\mathbb{D}$ for any constant C, then f has finite valence. In either case it would be interesting to find a sharp bound on the valence of f in terms of C. If $|\mathcal{S}f(z)| \le C$, we can adapt the proof of Theorem A to show that any pair of points $\alpha, \beta \in \mathbb{D}$ where $f(\alpha) = f(\beta)$ must satisfy $|\alpha - \beta| \ge \sqrt{2/C}\,\pi$. (In particular, if $|\mathcal{S}f(z)| \le \pi^2/2$, then $|\alpha-\beta| \ge 2$ and so f is univalent in $\mathbb{D}$, in accordance with Nehari's theorem.) The lower bound is established by comparing the associated differential equation $u'' + (\mathcal{S}f/2)u = 0$ with $v'' + (C/2)v = 0$, whose solutions have zeros with separation $\sqrt{2/C}\,\pi$. An explicit upper bound on the valence of f then follows by consideration of the optimal packing of disks of fixed radius $\sqrt{1/2C}\,\pi$ in the unit disk. Details are pursued in a subsequent paper [5], which gives some quantitative estimates on the valence.

§3. Valence of harmonic lifts.

We now turn to harmonic mappings and show how the preceding theorems on valence of analytic functions can be generalized to the lifts of harmonic mappings to their associated minimal surfaces. A planar harmonic mapping is a complex-valued harmonic function

$$f(z) = u(z) + iv(z)\,, \qquad z = x + iy\,,$$

defined on some domain $\Omega \subset \mathbb{C}$. If Ω is simply connected, the mapping has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in Ω and $g(z_0) = 0$ for some specified point $z_0 \in \Omega$. The mapping f is locally univalent if and only if its Jacobian $|h'|^2 - |g'|^2$ does not vanish. It is said to be orientation-preserving if $|h'(z)| > |g'(z)|$ in Ω, or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega(z)| < 1$ in Ω.

According to the Weierstrass–Enneper formulas, a harmonic mapping $f = h + \overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$ lifts locally to a minimal surface described by conformal parameters if and only if its dilatation has the form $\omega = q^2$ for some meromorphic function q. The Cartesian coordinates (U, V, W) of the surface are then given by

$$U(z) = \mathrm{Re}\{f(z)\}\,, \quad V(z) = \mathrm{Im}\{f(z)\}\,, \quad W(z) = 2\,\mathrm{Im}\left\{\int_{z_0}^{z} \sqrt{h'(\zeta)g'(\zeta)}\,d\zeta\right\}\,.$$

We use the notation

$$\widetilde{f}(z) = \big(U(z), V(z), W(z)\big)$$

for the lifted mapping from Ω to the minimal surface. The first fundamental form of the surface is $ds^2 = \lambda^2|dz|^2$, where the conformal metric is

$$\lambda = e^{\sigma} = |h'| + |g'|\,.$$

The Gauss curvature of the surface at a point $\widetilde{f}(z)$ for which $h'(z) \neq 0$ is

$$K = -\frac{1}{\lambda^2}\Delta(\log\lambda) = -\frac{4|q'|^2}{|h'|^2(1+|q|^2)^4}\,, \tag{6}$$

where Δ is the Laplacian operator. Further information about harmonic mappings and their relation to minimal surfaces can be found in the book [9].

For a harmonic mapping $f = h + \overline{g}$ with $|h'(z)| + |g'(z)| \neq 0$, whose dilatation is the square of a meromorphic function, we have defined [3] the *Schwarzian derivative* by the formula

$$\mathcal{S}f = 2\big(\sigma_{zz} - \sigma_z^2\big)\,, \tag{7}$$

where $\sigma = \log\big(|h'| + |g'|\big)$ and

$$\sigma_z = \frac{\partial\sigma}{\partial z} = \frac{1}{2}\left(\frac{\partial\sigma}{\partial x} - i\frac{\partial\sigma}{\partial y}\right)\,, \qquad z = x + iy\,.$$

If f is analytic, $\mathcal{S}f$ is the classical Schwarzian. If f is harmonic and ϕ is analytic, then $f \circ \phi$ is harmonic and

$$\mathcal{S}(f \circ \phi) = ((\mathcal{S}f) \circ \phi)\phi'^2 + \mathcal{S}\phi\,,$$

a generalization of the classical formula. With $h'(z) \neq 0$ and $g'/h' = q^2$, a calculation (*cf.* [3]) produces the expression

$$\mathcal{S}f = \mathcal{S}h + \frac{2\overline{q}}{1+|q|^2}\left(q'' - q'\,\frac{h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2 . \tag{8}$$

It must be emphasized that we are not requiring our harmonic mappings to be locally univalent. In other words, the Jacobian need not be of constant sign in the domain Ω. The orientation of the mapping may reverse, corresponding to a folding in the associated minimal surface. It is also possible for the minimal surface to exhibit several sheets above a point in the (u, v)–plane. Thus the lifted mapping $\widetilde{f}$ may be locally or globally univalent even when the underlying mapping f is not.

In earlier work [4] we obtained the following criterion for the lift of a harmonic mapping to be univalent.

Theorem B. *Let $f = h + \overline{g}$ be a harmonic mapping of the unit disk, with $e^{\sigma(z)} = |h'(z)| + |g'(z)| \neq 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\widetilde{f}$ denote the Weierstrass–Enneper lift of f to a minimal surface with Gauss curvature $K = K(\widetilde{f}(z))$ at the point $\widetilde{f}(z)$. Suppose that the inequality* (1) *holds for some Nehari function p. Then $\widetilde{f}$ is univalent in $\mathbb{D}$.*

Examples in [4] show that the univalence may fail if the right-hand side of (1) is replaced by $Cp(z)$ for some $C > 2$, but we now give harmonic analogues of Theorem A and Theorem 1. Recall that μ is defined by the limit (4).

Theorem 2. *Let $f = h + \overline{g}$ be a harmonic mapping of the unit disk, with $e^{\sigma(z)} = |h'(z)| + |g'(z)| \neq 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\widetilde{f}$ denote the Weierstrass–Enneper lift of f to a minimal surface with Gauss curvature $K = K(\widetilde{f}(z))$ at the point $\widetilde{f}(z)$. Suppose that*

$$|\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \leq C\,p(|z|)\,, \qquad z \in \mathbb{D}\,, \tag{9}$$

for some Nehari function p and some constant $C = 2(1+\delta^2)$, where $\delta > 0$. Then $\widetilde{f}$ is uniformly locally univalent. If $\mu = 1$, so that $p(x) = (1-x^2)^{-2}$, then $d(\alpha, \beta) \geq \pi/\delta$ for any pair of distinct points in $\mathbb{D}$ where $\widetilde{f}(\alpha) = \widetilde{f}(\beta)$. If $C\mu < 2$, then $\widetilde{f}$ has finite valence that is bounded independently of f.

The proof will rely on devices similar to those used in the proof of Theorem B. The first is a notion of Schwarzian derivative for curves in $\mathbb{R}^n$, due to Ahlfors [1]. For a sufficiently smooth mapping $\varphi : (a, b) \to \mathbb{R}^n$, Ahlfors defined

$$S_1\varphi = \frac{\langle \varphi', \varphi''' \rangle}{|\varphi'|^2} - 3\frac{\langle \varphi', \varphi'' \rangle^2}{|\varphi'|^4} + \frac{3}{2}\frac{|\varphi''|^2}{|\varphi'|^2}$$

and proved its invariance under postcomposition with Möbius transformations. Chuaqui and Gevirtz [6] recently used Ahlfors' Schwarzian to give a univalence criterion for curves. Here is their result:

Theorem C. *Let $P(x)$ be continuous on an interval (a,b), with the property that no nontrivial solution of the differential equation $u''+Pu=0$ has more than one zero. Let $\varphi:(a,b)\to\mathbb{R}^n$ be a curve of class C^3 with $\varphi'(x)\neq 0$ on (a,b). If $S_1\varphi(x)\leq 2P(x)$ on (a,b), then φ is univalent.*

Proof of Theorem 2. Suppose first that f satisfies (9) with $C=2(1+\delta^2)$ and $p(x)=(1-x^2)^{-2}$. It may be observed that for this choice of Nehari function the hypothesis is Möbius invariant. More precisely, if T is any Möbius self-mapping of the disk, then $F(\zeta)=f(T(\zeta))$ satisfies

$$|\mathcal{S}F(\zeta)|+e^{2\tau(\zeta)}|K(\widetilde{F}(\zeta))|\leq\frac{C}{(1-|\zeta|^2)^2}\,,\qquad \zeta\in\mathbb{D}\,, \tag{10}$$

where $\widetilde{F}=\widetilde{f}(T(\zeta))$ is the lift of the F to the same minimal surface, now endowed with the conformal metric $e^{\tau(\zeta)}=e^{\sigma(T(\zeta))}|T'(\zeta)|$. But

$$\mathcal{S}F(\zeta)=(\mathcal{S}f)(T(\zeta))T'(\zeta)^2\,,$$

by the transformation property of the Schwarzian, so (10) follows from (9) via the relation

$$\frac{|T'(\zeta)|}{1-|T(\zeta)|^2}=\frac{1}{1-|\zeta|^2}\,.$$

Now suppose that $\widetilde{f}(z_1)=\widetilde{f}(z_2)$ for some pair of distinct points $z_1,z_2\in\mathbb{D}$. Let T be the Möbius transformation of $\mathbb{D}$ onto itself such that $T(0)=z_1$ and $T(\xi)=z_2$ for some point ξ in the real interval $(0,1)$. Let $F=f\circ T$ and consider the curve $\varphi(x)=\widetilde{F}(x)$, for which $\varphi(0)=\varphi(\xi)$. A calculation (*cf.* [4], Lemma 1) shows that

$$S_1\widetilde{F}(x)\leq\mathrm{Re}\{\mathcal{S}F(x)\}+e^{2\tau(x)}|K(\widetilde{F}(x))|\,,\qquad -1<x<1\,.$$

Consequently, it follows from (10) that

$$S_1\varphi(x)\leq\frac{C}{(1-|\zeta|^2)^2}\,,\qquad -1<x<1\,.$$

This allows an application of Theorem C with $P(x)=(1+\delta^2)(1-x^2)^{-2}$. Recall that the differential equation (2) has a solution (3) that does not vanish in an interval $(0,b)$ whose endpoints are at hyperbolic distance $d(0,b)=\pi/\delta$. In view of the Sturm separation theorem, this implies that no nontrivial solution of (2) can vanish more than once in the interval $[0,b)$. Thus Theorem C applies to show that φ is univalent in $[0,b)$. But $\varphi(0)=\varphi(\xi)$ and so $d(0,\xi)\geq d(0,b)=\pi/\delta$. By the Möbius invariance of the hyperbolic metric, it follows that $d(z_1,z_2)\geq\pi/\delta$. (Strictly speaking, $0 \notin (0,b)$ and so Theorem C must be applied to a sequence of intervals (a_n, b_n) with $a_n < 0 < b_n$, $d(a_n, b_n) = \pi/\delta$, and $a_n \to 0$.)

For any other Nehari function $p(x)$, the property $\mu \le 1$ implies that $C\,p(|z|) \le C_1(1-|z|^2)^{-2}$ in $\mathbb{D}$, and so if (9) holds it follows from what we have just shown that $\widetilde{f}$ is uniformly locally univalent.

Suppose next that $C\mu < 2$. To conclude that $\widetilde{f}$ has bounded valence, we adapt the argument given by Gehring and Pommerenke [10] for analytic functions. First note that $C\mu < 2$ implies $\mu < 1$, and so by definition of μ it follows from (9) that

$$|\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \le \frac{C_1}{(1-|z|^2)^2} \tag{11}$$

in some annulus $r < |z| < 1$, for some constant $C_1 < 2$, where the radius r depends on C and p but not on f. Now let Ω be an "almost circular" domain in the annulus $r < |z| < 1$ whose boundary consists of a small arc of the unit circle together with an arc of a circle tangent to the circle $|z| = r$. Let ψ be a conformal mapping of $\mathbb{D}$ onto Ω, and define $F = f \circ \psi$. Then the Schwarzian of F is

$$\mathcal{S}F(\zeta) = (\mathcal{S}f)(\psi(\zeta))\psi'(\zeta)^2 + \mathcal{S}\psi(\zeta)\,, \qquad \zeta \in \mathbb{D}\,.$$

Explicit calculation (*cf.* [10], p. 238) shows that the Schwarzian norm $\|\mathcal{S}\psi\|$, as defined in (14) below, can be made arbitrarily small by choosing the domain Ω to be sufficiently circular. (Intuitively, this makes ψ "almost" a Möbius transformation, which has zero Schwarzian.) Furthermore, since ψ maps $\mathbb{D}$ into itself, it satisfies the inequality

$$\frac{|\psi'(\zeta)|}{1-|\psi(\zeta)|^2} \le \frac{1}{1-|\zeta|^2}\,, \qquad \zeta \in \mathbb{D}\,.$$

But the composite mapping $F = f \circ \psi$ has a lift $\widetilde{F}(\zeta) = \widetilde{f}(\psi(\zeta))$ to the same minimal surface with conformal metric

$$e^{\tau(\zeta)} = e^{\sigma(\psi(\zeta))}|\psi'(\zeta)|\,.$$

Combining these relations with the inequality (11), we find that

$$\begin{aligned} |\mathcal{S}F(\zeta)| + e^{2\tau(\zeta)}|K(\widetilde{F}(\zeta))| &\le \frac{C_1}{(1-|\zeta|^2)^2} + |\mathcal{S}\psi(\zeta)| \\ &\le \frac{C_2}{(1-|\zeta|^2)^2}\,, \qquad \zeta \in \mathbb{D}\,, \end{aligned}$$

where $C_1 < C_2 < 2$, if Ω is sufficiently circular. Invoking Theorem B with $p(x) = (1-x^2)^{-2}$, we conclude that $\widetilde{F}$ is univalent in $\mathbb{D}$. Therefore, $\widetilde{f}$ is univalent in Ω.

Now let $\rho = \frac{1}{2}(1+r)$ and observe that the annulus $\rho < |z| < 1$ is contained in the union of a finite number of rotated copies of Ω. Since $\widetilde{f}$ is univalent in each of these copies of Ω, it follows that $\widetilde{f}$ has finite valence in the annulus $\rho < |z| < 1$, and the valence has a bound independent of f. On the other hand, the uniform local univalence shows that $\widetilde{f}$ has finite and bounded valence in each closed subdisk $|z| \le R < 1$. Therefore, $\widetilde{f}$ has finite and bounded valence in $\mathbb{D}$, and the proof of Theorem 2 is complete.

§4. An example.

Our proof of Theorem A essentially incorporates the example given by Hille [11] to show that Nehari's original univalence criterion is best possible. We now adapt Hille's example to see that the inequality $d(\alpha,\beta)\geq\pi/\delta$ of Theorem 2 is best possible for harmonic mappings that lift to a catenoid.

The harmonic function $w=f(z)=z+1/\overline{z}$ maps the punctured plane $0<|z|<\infty$ onto the doubly covered annular region $2\leq|w|<\infty$, and it lifts to the mapping $\widetilde{f}=(U,V,W)$ of the punctured plane onto the catenoid defined by

$$U=(r+1/r)\cos\theta\,,\quad V=(r+1/r)\sin\theta\,,\quad W=2\log r\,,$$

where $z=re^{i\theta}$. Consider now the composition $F=f\circ\phi$, with

$$z=\phi(\zeta)=c\left(\frac{1+\zeta}{1-\zeta}\right)^{i\delta}=c\,\exp\left\{i\delta\log\frac{1+\zeta}{1-\zeta}\right\}\,,\qquad \zeta\in\mathbb{D}\,.$$

Here $\delta>0$ and $c>0$ are fixed parameters. Although $\widetilde{f}$ is univalent, the lifted mapping $\widetilde{F}=\widetilde{f}\circ\phi$ has infinite valence, since for instance ϕ maps the real segment $-1<\zeta<1$ onto the infinitely covered circle $|z|=c$. As in the proof of Theorem A, we find that $\widetilde{f}(x_n)=\widetilde{f}(x_{n+1})$ for a sequence of points x_n on the real axis at hyperbolic distance $d(x_n,x_{n+1})=\pi/\delta$.

On the other hand, calculations lead to the Schwarzian expressions

$$\mathcal{S}f(z)=\frac{4\,|z|^2}{z^2(1+|z|^2)^2}\,,\qquad \mathcal{S}\phi(\zeta)=\frac{2(1+\delta^2)}{(1-\zeta^2)^2}\,,$$

so the chain rule gives

$$\mathcal{S}F(\zeta)=\frac{2}{(1-\zeta^2)^2}\left\{1+\delta^2-\frac{8\,\delta^2\,|\phi(\zeta)|^2}{(1+|\phi(\zeta)|^2)^2}\right\}\,. \tag{12}$$

By the formula (6) with $h'(z)=1$ and $q(z)=i/z$, the Gauss curvature of the catenoid is found to be

$$K(\widetilde{f}(z))=-\frac{4\,|z|^4}{(1+|z|^2)^4}\,,$$

and so

$$e^{2\sigma(z)}K(\widetilde{f}(z))=-\frac{4}{(1+|z|^2)^2}\,.$$

Thus

$$\begin{aligned} e^{2\tau(\zeta)}|K(\widetilde{F}(\zeta))| &= e^{2\sigma(\phi(\zeta))}|\phi'(\zeta)|^2|K(\widetilde{f}(\phi(\zeta)))| \\ &= \frac{4\,|\phi'(\zeta)|^2}{(1+|\phi(\zeta)|^2)^2}=\frac{16\,\delta^2\,|\phi(\zeta)|^2}{(1+|\phi(\zeta)|^2)^2\,|1-\zeta^2|^2}\,. \end{aligned} \tag{13}$$

Finally, the expressions (12) and (13) combine to show that

$$|\mathcal{S}F(\zeta)| + e^{2\tau(\zeta)}|K(\widetilde{F}(\zeta))| = \frac{2(1+\delta^2)}{|1-\zeta^2|^2} \le \frac{2(1+\delta^2)}{(1-|\zeta|^2)^2}$$

if the constant c is chosen sufficiently small. In other words, the harmonic mapping F satisfies the hypothesis of Theorem 2 with $C = 2(1+\delta^2)$ and Nehari function $p(x) = (1-x^2)^{-2}$, yet $d(\alpha,\beta) = \pi/\delta$ at certain points where $\widetilde{F}(\alpha) = \widetilde{F}(\beta)$, so the general estimate $d(\alpha,\beta) \ge \pi/\delta$ cannot be improved in this case.

§5. Schwarzian norms of harmonic mappings.

Let $f = h + \overline{g}$ be a locally univalent orientation-preserving harmonic mapping with dilatation $g'/h' = q^2$, where q is analytic and $|q(z)| < 1$ in the unit disk $\mathbb{D}$. Then the Schwarzian $\mathcal{S}f$ is defined by (7) and has the expression (8). Let $\|\mathcal{S}f\|$ denote the hyperbolic norm of $\mathcal{S}f$; that is,

$$\|\mathcal{S}f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)^2 |\mathcal{S}f(z)| \,. \tag{14}$$

It should be observed that the Schwarzian norm is Möbius invariant. In other words, $\|\mathcal{S}(f\circ T)\| = \|\mathcal{S}f\|$ if T is any Möbius self-mapping of the disk.

If f is *analytic* and univalent in $\mathbb{D}$, it has long been known that $\|\mathcal{S}f\| \le 6$, a result due to Kraus [13]. The bound is sharp, since the Koebe function $k(z) = z/(1-z)^2$ has Schwarzian

$$\mathcal{S}k(z) = -\frac{6}{(1-z^2)^2}\,.$$

In view of this result for analytic functions, it is natural to ask whether every univalent harmonic mapping f has finite Schwarzian norm. We have not been able to settle this question, but we can show, with no assumption of global univalence, that a harmonic mapping has finite Schwarzian norm if and only if its analytic part does.

Theorem 3. *Let $f = h + \overline{g}$ be a locally univalent orientation-preserving harmonic mapping whose dilatation is the square of an analytic function in the unit disk. Then $\|\mathcal{S}f\| < \infty$ if and only if $\|\mathcal{S}h\| < \infty$. In particular, $\|\mathcal{S}f\| < \infty$ if h is univalent.*

Proof. Suppose first that $\|\mathcal{S}h\| < \infty$. By hypothesis, f has dilatation $g'/h' = q^2$ for some analytic function q with $|q(z)| < 1$. Hence

$$\frac{2\,|q(z)|}{1+|q(z)|^2} \le 1\,, \qquad z \in \mathbb{D}\,,$$

and so it follows from (8) that

$$|\mathcal{S}f(z)| \le |\mathcal{S}h(z)| + |q''(z)| + |q'(z)| \left|\frac{h''(z)}{h'(z)}\right| + |q'(z)|^2\,.$$

By the Schwarz–Pick lemma,

$$|q'(z)| \leq \frac{|q'(z)|}{1-|q(z)|^2} \leq \frac{1}{1-|z|^2}\,.$$

If a function ϕ is analytic in the unit disk and $|\phi(z)| \leq 1/(1-|z|^2)$, then it follows from Cauchy's integral formula that $|\phi'(z)| \leq 4/(1-|z|^2)^2$. We apply this to the function $\phi = q'$ to see that

$$|q''(z)| \leq \frac{4}{(1-|z|^2)^2}\,.$$

Finally, a result of Pommerenke ([20], p. 133) asserts that

$$(1-|z|^2)\left|\frac{h''(z)}{h'(z)}\right| \leq 2 + 2\bigl(1+\tfrac{1}{2}\|\mathcal{S}h\|\bigr)^{1/2}\,. \tag{15}$$

Putting the estimates together, we conclude that

$$\|\mathcal{S}f\| \leq \|\mathcal{S}h\| + 2\bigl(1+\tfrac{1}{2}\|\mathcal{S}h\|\bigr)^{1/2} + 7\,.$$

Conversely, suppose that $\|\mathcal{S}f\| < \infty$. The formula (8) and the preceding estimates show that

$$|\mathcal{S}h(z)| \leq |\mathcal{S}f(z)| + \frac{5}{(1-|z|^2)^2} + \frac{1}{1-|z|^2}\left|\frac{h''(z)}{h'(z)}\right|\,. \tag{16}$$

In order to use Pommerenke's estimate of h''/h', we apply the inequality (16) to the dilated function $f_r = h_r + \overline{g_r}$, where $0 < r < 1$ and $f_r(z) = f(rz)$. Note that $\mathcal{S}f_r(z) = r^2\mathcal{S}f(rz)$, so that

$$(1-|z|^2)^2|\mathcal{S}f_r(z)| \leq (1-|rz|^2)^2|\mathcal{S}f(rz)| \leq \|\mathcal{S}f\|\,.$$

Because $\|\mathcal{S}h_r\|$ is finite for each $r < 1$, we can apply (16) to f_r and invoke (15) to infer that

$$(1-|z|^2)^2|\mathcal{S}h_r(z)| \leq \|\mathcal{S}f\| + 7 + 2\bigl(1+\tfrac{1}{2}\|\mathcal{S}h_r\|\bigr)^{1/2}\,,$$

or

$$\|\mathcal{S}h_r\| - 2\bigl(1+\tfrac{1}{2}\|\mathcal{S}h_r\|\bigr)^{1/2} \leq \|\mathcal{S}f\| + 7\,.$$

Now let $r \to 1$ to conclude that $\|\mathcal{S}h\| < \infty$.

Finally, we observe that univalent harmonic mappings with range convex in one direction have finite Schwarzian norm. These mappings are obtained by a known process of shearing conformal mappings whose range is convex in one direction. (See [9], Section 3.4 for the shear construction.)

Theorem 4. *Suppose a function ϕ is analytic and univalent in the unit disk, and its range is convex in the horizontal direction. Let $f = h + \overline{g}$ be the harmonic shear of ϕ in the horizontal direction with dilatation q^2, where q is analytic and $|q(z)| < 1$ in the disk. Then $\|\mathcal{S}f\| < \infty$.*

Proof. As shown in [3], the Schwarzian of f has the form

$$\mathcal{S}f = \mathcal{S}\phi + \frac{2({q'}^2 + (1-q^2)qq'')}{(1-q^2)^2} - \frac{2qq'}{1-q^2}\frac{\phi''}{\phi'} + \frac{2\,\overline{q}}{1+|q|^2}\left\{q'' - q'\left(\frac{\phi''}{\phi'} + \frac{2qq'}{1-q^2}\right)\right\} - 4\left(\frac{\overline{q}\,q'}{1+|q|^2}\right)^2.$$

The preceding estimates for q and its derivatives can now be applied to derive the inequality

$$|\mathcal{S}f(z)| \le |\mathcal{S}\phi(z)| + 2\left|\frac{q''(z)}{1-q(z)^2}\right| + \frac{3}{1-|z|^2}\left|\frac{\phi''(z)}{\phi'(z)}\right| + \frac{9}{(1-|z|^2)^2}\,.$$

Since $|q'(z)/(1-q(z)^2)| \le 1/(1-|z|^2)$, Cauchy's integral formula shows that the derivative satisfies

$$\left|\frac{q''(z)}{1-q(z)^2} + \frac{2q(z)q'(z)^2}{(1-q(z)^2)^2}\right| \le \frac{4}{(1-|z|^2)^2}\,,$$

which implies that

$$\left|\frac{q''(z)}{1-q(z)^2}\right| \le \frac{6}{(1-|z|^2)^2}\,.$$

Since ϕ is an analytic univalent function, a standard inequality (*cf.* [8], p. 32) shows that

$$\left|\frac{\phi''(z)}{\phi'(z)}\right| \le \frac{6}{1-|z|^2}\,,$$

and Kraus' theorem gives $(1-|z|^2)^2|\mathcal{S}\phi(z)| \le 6$. Combining these estimates, we find that

$$\|\mathcal{S}f\| \le 6 + 12 + 18 + 9 = 45.$$

The harmonic shear of the Koebe function with dilatation z^2 is $f = h + \overline{g}$, where

$$h(z) = \frac{\frac{1}{3}}{(1-z)^3} \qquad \text{and} \qquad g(z) = \frac{z^2 - z + \frac{1}{3}}{(1-z)^3}\,,$$

up to additive constants. Its Schwarzian is found to be

$$\mathcal{S}f(z) = -4\left(\frac{1}{1-z} + \frac{\overline{z}}{1+|z|^2}\right)^2.$$

(See [3], p. 337, where the formula is printed incorrectly.) Note that $\|\mathcal{S}f\| < \infty$, in agreement with Theorem 4. Although h is not univalent, it has Schwarzian $\mathcal{S}h(z) = -4/(1-z)^2$ and Schwarzian norm $\|\mathcal{S}h\| = 16$. Since $\|\mathcal{S}h\| < \infty$, the finiteness of $\|\mathcal{S}f\|$ is also in agreement with Theorem 3.

Addendum. In subsequent work [5] we have been able to show that the Schwarzian norms of univalent harmonic mappings are finite and have the uniform bound $\|\mathcal{S}f\| < 19,407$.

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